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A Limit Theorem of Discretization with Weights Large on Nodes

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By combining discretization and weighting on nodes, one can in the limit approximate on infinite sets under Lagrange-type interpolatory constraints, enabling the use of existing algorithms and programs.

Let W be a compact metric space with metric ρ. For any compact subset Y of W, let $C(Y)$ be the space of real (or complex) continuous functions on Y and for $g \in C(Y)$, define

$$\|g\|_Y = \sup\{|g(x)|: x \in Y\}.$$

Let X be a compact subset of W and Z a finite subset of X. Let F be a given approximating function with parameter A taken from a nonempty closed subset P of real (or complex) n-space such that $F(A, \cdot) \in C(W)$ for all $A \in P$. The problem of approximation on X with interpolation on Z is: given $f \in C(W)$, find a parameter $A^* \in P$ minimizing $\|f - F(A, \cdot)\|_X$ over A subject to the constraint

$$F(A, x) = f(x), \qquad x \in Z. \tag{*}$$

Such a parameter A^* is called best, and $F(A^*, \cdot)$ is called a best approximation to f on X with interpolation on Z.

Let $\| \ \|_c$ be the maximum norm on n-space.

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DEFINITION. A compact subset Y of W is called *parameter bounding* if for a sequence $\{A^k\} \subset P$, $\|A^k\|_c \to \infty$ implies $\|F(A^k, \cdot)\|_Y \to \infty$.

DEFINITION. (F, P) satisfies *Young's condition* [3, 4] if

(i) W has a parameter bounding subset for (F, P),

(ii) $A \in P$, $\{A^k\} \subset P$, $\{A^k\} \to A$ implies $\{F(A^k, \cdot)\} \to F(A, \cdot)$ uniformly on W.

Families with Young's condition include finite-dimensional linear families, real families unisolvent on an interval $[a, b]$, tame rationals [5, 6], and some transformations thereof [3, 6].

Dunham [2] has shown that interpolating approximation by families satisfying Young's condition is the limit of weighted approximation with weights 1 off Z and tending to ∞ on Z, and that [3] approximation by families satisfying Young's condition (without interpolation) on compact X is the limit of approximation on X_k, where $\{X_k\} \to X$.

DEFINITION [7]. Let X, Y be nonempty subsets of W, define

$$\operatorname{dist}(X, Y) = \sup\{\inf[\rho(x, y): x \in X]: y \in Y\},$$

$$d(X, Y) = \max\{\operatorname{dist}(X, Y), \operatorname{dist}(Y, X)\}.$$

Let $X, X_1, X_2, \ldots, X_k, \ldots$ be compact subsets of W. We say $\{X_k\} \to X$ if $d(X_k, X) \to 0$.

Combining the results of [2] and [3] (with a slight generalization [7] of the definition of $\{X_k\} \to X$ given in [3]), if we approximate on X_k tending to X with a weight w_k which is one off Z and which tends to ∞ on Z, the limit would be best on X with respect to $(*)$. Best approximations with respect to weights w_k exist by standard arguments.

THEOREM. *Let (F, P) satisfy Young's condition. Let $\{X_k\} \to X$, $Z \subset X_k$, $Y \subset X \cap X_k$, where Y is a parameter bounding set. Let there exist $F(B, \cdot)$ satisfying $(*)$. Let $\{w_k\}$ be a sequence of positive weight functions on W such that $w_k = 1$ off Z and $w_k(x) \to \infty$ for $x \in Z$. Let A^k be best on X_k with respect to w_k. Then $\{A^k\}$ has an accumulation point and if A^0 is an accumulation point, A^0 is best and there is a sequence $\{k(j)\}$ such that $\{F(A^{k(j)}, \cdot)\} \to F(A^0, \cdot)$ uniformly on W.*

Proof. For convenience, the norm on X_k will be denoted by $\| \ \|_k$ and the norm on X by $\| \ \|$. Suppose $\{A^k\}$ is unbounded. From Young's condition, $\{\|F(A^k, \cdot)\|_Y\}$ is unbounded and, as $Y \subset X_k$, $\{\|F(A^k, \cdot)\|_k\}$ is unbounded. Hence $\{\|f-F(A^k, \cdot)\|_k\}$ is unbounded, $\{\|w_k(f-F(A^k, \cdot))\|_k\}$ is unbounded. But

$$\begin{aligned}\|w_k \cdot (f-F(A^k, \cdot))\|_k &\leqslant \|w_k \cdot (f-F(B, \cdot))\|_k \\ &= \|f-F(B, \cdot)\|_k \\ &\leqslant \|f-F(B, \cdot)\|_W < \infty,\end{aligned}$$

this is a contradiction.

As $\{A^k\}$ is bounded, it has an accumulation point A^0. By taking a subsequence if necessary, we can assume $\{A^k\} \to A^0$. We claim A^0 satisfies $(*)$. Suppose not then there is $\varepsilon > 0$ and $x \in Z$ such that $|f(x) - F(A^0, x)| > \varepsilon$. Since $F(A^k, x) \to F(A^0, x)$, we have for k sufficiently large $|f(x) - F(A^k, x)| > (\varepsilon/2)$, hence

$$\|w_k(f-F(A^k, \cdot))\|_k \geqslant w_k(x)|f(x) - F(A^k, x)| \to \infty;$$

this is a contradiction again.

Now we prove

$$\liminf_{k \to \infty} \|w_k(f-F(A^k, \cdot))\|_k \geqslant \|f-F(A^0, \cdot)\| . \tag{1}$$

Let $x \in X$ such that $\|f-F(A^0, \cdot)\| = |f(x) - F(A^0, x)|$. As $\{X_k\} \to X$, there exist $x_k \in X_k$, $\{x_k\} \to x$. Then

$$\begin{aligned}|f(x) - F(A^0, x)| &\leqslant |f(x) - f(x_k)| + |f(x_k) - F(A^k, x_k)| \\ &\quad + |F(A^k, x_k) - F(A^0, x_k)| + |F(A^0, x_k) - F(A^0, x)|, \\ |f(x_k) - F(A^k, x_k)| &\geqslant \|f-F(A^0, \cdot)\| - |f(x) - f(x_k)| \\ &\quad - |F(A^k, x_k) - F(A^0, x_k)| \\ &\quad - |F(A^0, x_k) - F(A^0, x)|.\end{aligned}$$

For given $\varepsilon > 0$, there exists K such that, for $k > K$,

$$|f(x) - f(x_k)| + |F(A^k, x_k) - F(A^0, x_k)| + |F(A^0, x_k) - F(A^0, x)| < \varepsilon.$$

Hence, for $k > K$,

$$\begin{aligned}\|w_k(f-F(A^k, \cdot))\|_k \geqslant \|f-F(A^k, \cdot)\|_k &\geqslant |f(x_k) - F(A^k, x_k)| \\ &> \|f-F(A^0, \cdot)\| - \varepsilon,\end{aligned}$$

and (1) is proven.

Suppose A^0 is not best with respect to $(*)$. Then there is C satisfying $(*)$ and $\varepsilon > 0$ such that

$$\|f - F(C, \cdot)\| < \|f - F(A^0, \cdot)\| - \varepsilon. \tag{2}$$

From (1), for k sufficiently large,

$$\|w_k(f - F(A^k, \cdot))\|_k > \|f - F(A^0, \cdot)\| - \varepsilon/2. \tag{3}$$

Let $x_k \in X_k$ such that $\|f - F(C, \cdot)\|_k = |f(x_k) - F(C, x_k)|$. As $|f(x) - F(C, x)|$ is uniformly continuous on W, and $\{X_k\} \to X$, for k sufficiently large, there exist $y_k \in X$ such that

$$|f(x_k) - F(C, x_k)| < |f(y_k) - F(C, y_k)| + \varepsilon/2,$$

hence for k sufficiently large,

$$\|f - F(C, \cdot)\|_k < \|f - F(C, \cdot)\| + \varepsilon/2. \tag{4}$$

We note that inequality (4) is valid for any continuous function on W. From (2)–(4), we have for k sufficiently large,

$$\|w_k(f - F(A^k, \cdot))\|_k > \|f - F(C, \cdot)\|_k. \tag{5}$$

But $\|f - F(C, \cdot)\|_k = \|w_k(f - F(C, \cdot))\|_k$, (5) contradicts optimality of A^k. A^0 is best, and uniform convergence follows by Young's condition.

Remark. If f has a unique best interpolating approximation $F(A^0, \cdot)$, we have $\{F(A^k, \cdot)\} \to F(A^0, \cdot)$ uniformly on W (even if A^0 is not unique). In fact, suppose not, then $\{F(A^k, \cdot)\}$ has at least two limit points.

Remark. The conclusion may not hold if Young's condition fails (see the example at the end of [7]).

Remark. We have

$$\lim_{k \to \infty} \|w_k(f - F(A^k, \cdot))\|_k = \|f - F(A^0, \cdot)\|. \tag{6}$$

In fact,

$$\|w_k(f - F(A^k, \cdot))\|_k \leqslant \|w_k(f - F(A^0, \cdot))\|_k = \|f - F(A^0, \cdot)\|_k,$$

and, by the note after inequality (4), given $\varepsilon > 0$, for k sufficiently large,

$$\|f - F(A^0, \cdot)\|_k < \|f - F(A^0, \cdot)\| + \varepsilon.$$

Hence,

$$\limsup_{k \to \infty} \|w_k(f - F(A^k, \cdot))\|_k \leqslant \|f - F(A^0, \cdot)\|. \tag{7}$$

(1) and (7) imply (6).

The results of this paper are of practical interest as programs and algorithms for weighted discrete approximation are available [1, Chap. 2, 4, pp. 173–176, 5, p. 21ff, 6, pp. 9–10, 8, 9, 10].

Remark. As noted in [10, p. 142], *general* linear or rational approximation can absorb weights even if the problem statement or program makes no mention of them), but are scarce for approximation with interpolation. For example, if X is an interval $[a, b]$, we can choose $X'_k \cup Z$ as X_k, where X'_k is a discrete set of $k+1$ equally spaced points on $[a, b]$, whose endpoints are included in X'_k.

In contrast, merely increasing weights on nodes [2] on infinite X (e.g., X an interval) does not yield an algorithm, as the weights need not be continuous on X.

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